PHYSICAL REVIEW E VOLUME 61, NUMBER 6 JUNE 2000

Statistical characterization of random electrostatic potentials

M. Aldana, H. Larralde, and G. Martínez-Mekler
Centro de Ciencias Físicas, UNAM, Apartado Postal 48-3, Cuernavaca, Morelos, CP 62251, Mexico

(Received 14 January 2000)

In this work we study statistical properties of random electrostatic potentials generated by one dimensional lattices with random charges. We show that the resulting random potentials are correlated Gaussian processes, satisfying the Lindeberg version of the central limit theorem, if certain restrictions are imposed on the individual potentials generated by the particles on the lattice. Since most of the point-particle electrostatic potentials occurring in nature satisfy the Lindeberg condition, the correlation properties of the random potentials are not arbitrary and must comply with the central limit theorem. Based on this theorem we can obtain explicit expressions for these correlations. We thus are able to give a characterization of a broad class of potentials yielding feasible physical scenarios. We illustrate some consequences of our findings by considering dynamical properties of a test particle interacting with the lattice. We show how the long range correlations generate statistical features in these properties, which are best exhibited when considering different length scales.

PACS number(s): 05.40.−a, 02.50.Ey

I. INTRODUCTION

Random potentials have long been an integral part of the modeling of disordered systems [1]. They appear in the description of localization phenomena, anomalous transport, pinning, glassy states, intracellular transport, molecular ratchets, and in many other fields of physics, chemistry and biology. However, despite this wide range of applicability of random potentials, their statistical properties are usually postulated as part of the model, with little or no emphasis on whether a potential landscape with the proposed properties can be physically generated or not [2,3].

It has also become clear by now that spatial correlations of the potentials play a crucial role in the processes under investigation. In many physical systems, these correlations are present because realistic potentials allow particles to interact at a distance. Even a completely uncorrelated distribution of random charges will give rise to a spatially correlated potential landscape.

In this work we study the statistical properties of the electrostatic potential generated by a set of random charges arranged on a one dimensional lattice, interacting with a test particle placed a distance away. Our main purpose is to give a statistical characterization of the interaction potential, and to present a simple formalism to deal with its joint probability distribution function. In doing so, we shall see that most of the electrostatic potentials occurring in nature satisfy the hypothesis of the central limit theorem, and therefore not every imaginable spatial correlation of the random potentials may be physically achievable. As an example, we study the dynamics of an overdamped test particle forced to move along the random potential. By analyzing the Fourier spectrum of the particle's velocity, we show spatial statistical features and scaling properties arising from the correlations of the potential.

The system that we shall study is shown schematically in Fig. 1, where a charged test particle Q interacts with a one dimensional lattice composed of n charged point particles $(n\to\infty)$ separated by a constant distance $d=1$. We can think of this one dimensional lattice as a very large polymer whose monomers are precisely the lattice particles. Thus, from now on we will refer to the lattice as the ''polymer'' and to the lattice particles as the ''monomers.''

The charge Q can move along the polymer, but it is constrained to remain at a constant distance σ perpendicular to the polymer; under these circumstances, the dynamics of the particle is one dimensional. The charge q_k of each monomer is either a discrete or continuous random variable whose probability density function $P(q)$ is the same for all the monomers. It is important to mention that by ''charged particle'' we do not mean just Coulombic particles. Both the charge Q and each one of the monomer charges q_k could be, instead, particles having permanent electric dipoles, polarizable particles with induced electric dipoles, molecules interacting through van der Waals forces, etc. As we shall see later, the potential experienced by the particle Q will be a Gaussian process for most physically relevant interaction potentials.

In the next section we discuss the type of electrostatic random potential generated by the system shown in Fig. 1. In our calculations we consider general point-particle electrostatic potential as a concrete example of the kind of potentials we will be working with, and the parameters involved in their specification. The results presented here can be generalized to a wider class of interaction potentials, as will be done in Sec. IV, where we consider ''screened'' potentials.

In Sec. III we analyze the integrability properties of the

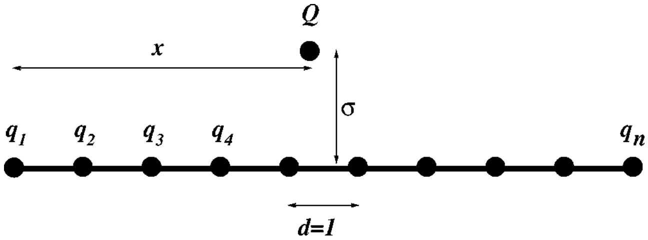

FIG. 1. Diagram illustrating the particle polymer basic setup: $d=1$ is the distance between the monomers, x is the position of the particle Q measured from the polymer origin, and σ is the perpendicular distance from the particle to the polymer.

individual potentials generated by each one of the monomers in the polymer, and discuss how these properties lead to different types of statistical characterizations. In Sec. IV we state the conditions that these individual potentials need to fulfill in order to have a statistical description based on the central limit theorem. If the Lindeberg condition holds, the overall random potential (i.e., the potential generated by the whole polymer) will be a Gaussian processes. In this section we also show that the electrostatic potentials most commonly occurring in nature satisfy the Lindeberg condition. Once we have ensured that the random potential is a correlated Gaussian process, we proceed to find the parameters characterizing its probability distribution. In Sec. V we present a simple formalism to compute the two-point joint probability distribution of the random potential explicitly. We should mention that our framework and many formal results parallel those of Rice [4] in the context of noise. This is to be expected as, after all, we are also considering an additive process. Nevertheless, as we are dealing with potentials, for which classically only the differences are physically relevant, we impose different and much less stringent restrictions on our elements of description. Also, we make a detailed classification of the statistics of the random potential in terms of the nature of the physical entities from which the potential arises.

In Sec. VI we illustrate how our results may be of use in the study of the dynamics of the particle along the polymer. We show that the velocity of the particle presents spatial statistical features that are a direct consequence of the long range correlation of the individual potentials. These features are exhibited in the power spectrum of the velocity, and are related to the scale invariance of the statistical structure of the potentials. Finally, we conclude this work in Sec. VII with a brief summary and discussion of our results, and present a classification of potentials in terms of the Lindeberg condition.

II. RANDOM POTENTIALS

Commonly, the interaction potential $U(r)$ between two charged particles q_1 and q_2 separated by a distance r_{12} can be represented by means of a potential law in the following way [5]:

$$U(r)=K\frac{q_1q_2}{r_{12}^{\alpha}}, \tag{1}$$

where K is a constant. The exponent α tell us what kind of interaction we are dealing with: $\alpha=1$ corresponds to a Coulomb interaction, $\alpha=2$ a dipolar interaction, $\alpha=3$ an ion-induced dipole interaction, and so on. We should keep in mind that by q_1 and q_2 we do not mean just Coulombic charges. They could also be dipolar moments, electrical polarizabilities, etc.

In the problem shown schematically in Fig. 1 we suppose that all the monomer charges $\{q_k\}$ are of the same physical type (e.g., they are all ions, or dipoles, etc.), and that the only thing changing from one charge to another is its value, which is an independent random variable with probability distribution function $P(q)$. The interaction potential between the kth-monomer with charge q_k on the polymer and the test particle Q is then given by

$$U_{\sigma,k}^{\alpha}(x)=a_k g_{\sigma}^{\alpha}(x-k), \tag{2}$$

where $a_k=KQq_k$ and $g_{\sigma}^{\alpha}(y)$ is defined as

$$g_{\sigma}^{\alpha}(y)\equiv\frac{1}{(y^2+\sigma^2)^{\alpha/2}}. \tag{3}$$

By summing up the contribution of each one of the charges on the polymer, we obtain the overall interaction potential $V(x)$ between the polymer and the test charge Q, namely,

$$V(x)=\sum_{k=-\infty}^{\infty}U_{\sigma,k}^{\alpha}(x)=\sum_{k=-\infty}^{\infty}a_k g_{\sigma}^{\alpha}(x-k) \tag{4}$$

(we have assumed that the polymer of infinite length). Note that the variables a_k have the same probability distribution $P(q)$ except for a constant normalization factor which can always be taken as unity. Therefore, from now on we shall make no distinction between q_k and a_k.

For our analysis we shall require the probability distribution $P(a)$ associated with the variables a_k to satisfy the following conditions: (a) $\langle a_k\rangle=\int_{-\infty}^{+\infty}aP(a)da=0$. (b) $\langle a_i a_k\rangle=\langle a_i\rangle\langle a_k\rangle=0$. (c) $P(a)=0$ if $|a|>R$, for some real number $R>0$.

In all other aspects, the probability distribution $P(a)$ is arbitrary. Condition (a) is not essential and is introduced just to simplify calculations; physically, it corresponds to appropriately setting the zero of the potential energy. Condition (b) establishes that the monomer charges q_k along the polymer are statistically independent one from the other, a property which is clearly inherited by the variables a_k [6]. Finally, condition (c) ensures that we are considering finite charges. In the next section we shall see that, for the convergence of Eq. (4), in some cases this condition can be relaxed by requiring only that $P(a)$ has a well defined second moment. Nevertheless, when we analyze the dynamics of the Q particle along the polymer, we shall require that a_k be bounded.

At this point, it is worth mentioning that the parameter σ, besides being the perpendicular distance from the polymer to the test particle, can be considered as the length scale at which the individual potentials $U_{\sigma,k}^{\alpha}(x)$ contribute appreciably to the overall interaction potential $V(x)$ at a particular position x [or, in other words, σ is the length scale at which the function $g_{\sigma}^{\alpha}(y)$ is appreciably different from zero]. In Fig. 2 we depict three graphs of the potential given by expression (4) for different values of σ, and $\alpha=4$. These graphs were obtained by allowing the variables a_k to take the values $\{\pm 1,\ \pm 2,\ \pm 3\}$ with the same probability [throughout this work we shall use this distribution of charges since we will see that some dynamical properties of the particle do not depend on the specific statistical nature of the charges a_n provided the probability distribution $P(a)$ satisfies conditions (a), (b), and (c) mentioned above]. We can observe from this figure that the potential landscape becomes smooth as σ increases (note the scale differences in the graphs). This smoothness of the potential as σ takes larger values is a consequence of the long range correlations among the individual potentials $U_{\sigma,k}^{\alpha}(x)$, i.e., it is due to the fact that when

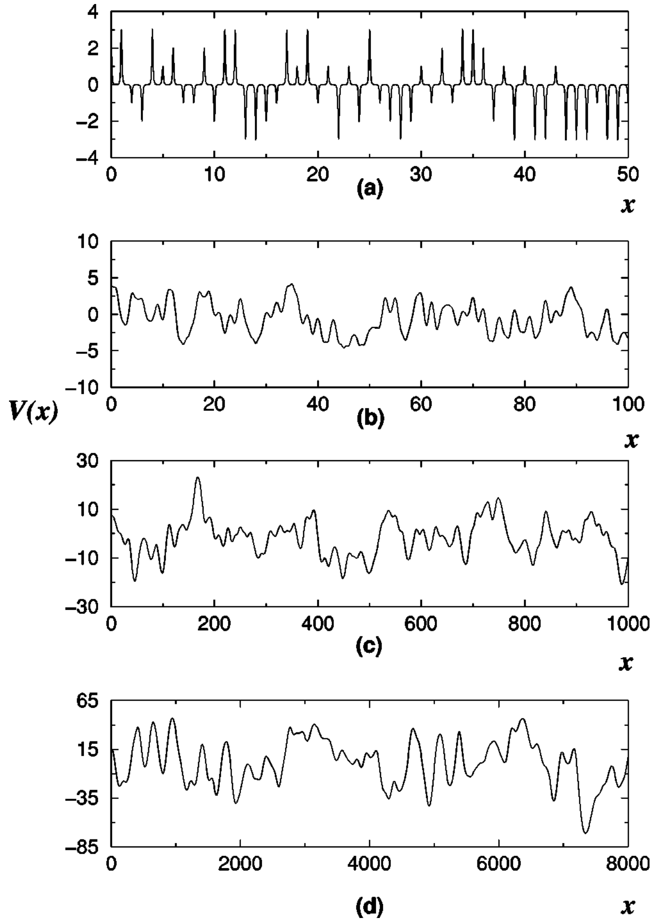

FIG. 2. Interaction potential between the particle and the polymer for successively increasing values of σ, and $\alpha=4$, using a probability distribution of charges for which the variables a_k take the values $\pm1,\pm2,\pm3$ with the same probability. (a) $\sigma=0.1$. (b) $\sigma=1$. (c) $\sigma=10$. (d) $\sigma=100$. In all these cases the amplitude of the potential has been scaled by a factor σ^{α}. Note that the interaction potential becomes smooth as σ increases.

σ increases, more and more monomers on the polymer contribute to the overall potential at a particular position x. The latter can be expressed by saying that, when the particle Q moves away, it is unable to ''see'' the fine structure of the potential, resolving only the coarse structure, while if the particle is very near to the polymer, it ''feels'' only the monomer just below it.

III. INTEGRABILITY PROPERTIES

It is clear that at a fixed position x, the overall interaction potential $V(x)$ is the sum of a very large number of independent random variables $\{U_{\sigma,k}^{\alpha}(x)\}$ each of them having its own probability distribution. We should then expect that in some appropriate limit the central limit theorem could be applied in order to find the r-point joint probability distribution $\mathcal{P}(V(x_1),V(x_2),\ldots,V(x_r))$ which statistically characterizes the potential $V(x)$. For this to happen it is necessary for the sum in Eq. (4) to remain finite for every value of x. Of course, the potentials $U_{\sigma,k}^{\alpha}(x)$ given in Eqs. (2) and (3) satisfy the above condition as long as $\alpha>1$, but in general, for other kinds of interaction potentials, this may not be the case.

A random potential $V(x)$ that can be written in the form

$$V(x)=\sum_{k=-\infty}^{\infty}a_k g(x-k), \tag{5}$$

where the a_k's are random variables and $g(y)$ is a given function, will be meaningful or not according to the integrability properties of $g(y)$. For our purposes, it will be convenient to distinguish between the three following cases.

(1) $\int_{-\infty}^{\infty}|g(y)|dy<\infty$. In this case $g(y)$ is absolutely integrable and no problems arise, since the sum in Eq. (5) remains finite, and therefore $V(x)$ is well defined for every x. Indeed, it is easy to see that $V(x)$ is bounded, since $|V(x)|=|\sum_{k=-\infty}^{\infty}a_k g(x-k)|\leqslant\max|a_k|\sum_{k=-\infty}^{\infty}|g(x-k)|\leqslant R\int_{-\infty}^{\infty}|g(y)|dy$, and we are supposing that this integral converges. We shall refer to this situation as the *bounded potential case*.

(2) $\int_{-\infty}^{\infty}|g(y)|dy$ does not converge, but $\int_{-\infty}^{\infty}[g(y)]^2dy<\infty$. Given that $\langle a_k\rangle=0$, the random potential $V(x)$ does not diverge at any position x but, since $g(y)$ is not absolutely integrable, $V(x)$ can acquire very large values. However, $V(x)$ has a typical value given by its variance, which is finite as can be seen:

$$\begin{aligned}\mathrm{Var}[V(x)]&=\langle[V(x)]^2\rangle\\&=\left\langle\left(\sum_{k=-\infty}^{\infty}a_k g(x-k)\right)^2\right\rangle\\&=\left\langle\sum_{k=-\infty}^{\infty}a_k g(x-k)\sum_{i=-\infty}^{\infty}a_i g(x-i)\right\rangle\\&=\sum_{k=-\infty}^{\infty}\langle a_k^2\rangle[g(x-k)]^2\\&\quad+\sum_{i\neq k}\langle a_i a_k\rangle g(x-i)g(x-k)\\&=\Delta^2\sum_{k=-\infty}^{\infty}[g(x-k)]^2\\&\approx\Delta^2\int_{-\infty}^{\infty}[g(y)]^2dy<\infty.\end{aligned}$$

where $\Delta^2=\langle a_k^2\rangle$ is the second moment of $P(a)$. To obtain this result we have made use of the properties (a) and (b) imposed on $P(a)$ in the preceding section, and instead of property (c), we only require the existence of Δ^2. Since in this case the random potential has a typical value, we will refer to this situation as the *localized potential case*.

(3) $\int_{-\infty}^{\infty}[g(y)]^2dy$ does not converge. If $g(y)$ is not square integrable the sum in Eq. (5) leads to divergences. This is due to the fact that the individual potentials $U_k(x)=a_k g(x-k)$ decay so slowly that the contributions coming from the charge fluctuations in the system do not vanish fast enough nor cancel out efficiently. This can be easily visualized in the simple case in which $g(x)$ is constant and the superposition of n of these individual potentials generated by the random charges yields the same ''random walk'' at every point in space, whose value typically diverges as the square root of n.

This case can be further characterized in two ways, corresponding to different physical behaviors.

Rough potential. In many instances, the quantities of physical interest are the forces derived from the potentials. If these forces are square integrable, then the central limit theorem can be applied to them, instead of to the potentials. Equation (5) is then substituted by an analogous one involving the derivatives of the function $g(y)$. Also, there are situations in which the potential landscape is of particular relevance to the physical problem, as would be the case in Brownian transport and other escape rate related phenomena. In these situations, the size and distribution of the potential wells and barriers are the crucial elements for a description. Hence, we are interested in characterizing the statistical properties of ''potential differences,'' and we can do this by considering as the random variables involved in the description the quantities $W_k(x)$ defined as $\{W_k(x)=U_k(x)-U_k(y),\ \mathrm{k}=1,2,3,\dots\}$, which is equivalent to setting the potential to zero at the arbitrary position y. If square integrability holds for $W_k(x)$, we can again invoke the central limit theorem for these quantities. In general, a property arising under such circumstances is that the resulting random potential is no longer ''regular,'' but rather it becomes rough at large length scales in the sense that $\langle[V(x_1)-V(x_2)]^2\rangle \sim |x_1-x_2|^\nu$, where ν is a positive scaling exponent having to do with the specific type of interaction we are considering.

Nondifferentiable potentials. When neither the potential derivatives nor the potential differences are square integrable, the resulting random potential is nowhere differentiable, which is not physically meaningful. The above occurs when the individual potentials increase fast enough with distance.

It is clear so far that in order to perform a statistical analysis of the random potential $V(x)$ based upon the central limit theorem it is necessary for the individual potentials $U_k(x)$ to be bounded or localized. Nevertheless, this is a necessary condition, but not a sufficient one, as we will see in the next section, where we show that another way for the central limit theorem to fail is if the range of interaction is extremely narrow. For the time being, we should mention that the class of potentials $U^\alpha_{\sigma,k}(x)$ given in Eqs. (2) and (3) give rise to three of the cases mentioned above, depending on the value of α: if $\alpha>1$ we are in the bounded potential case, while if $1/2<\alpha\leqslant 1$ we have the localized potential case and if $-1/2<\alpha\leqslant 1/2$ the rough potential case. In what follows, we shall always assume that the interaction potentials we are dealing with are bounded or localized [i.e., $g(y)$ is square integrable].

IV. CENTRAL LIMIT THEOREM

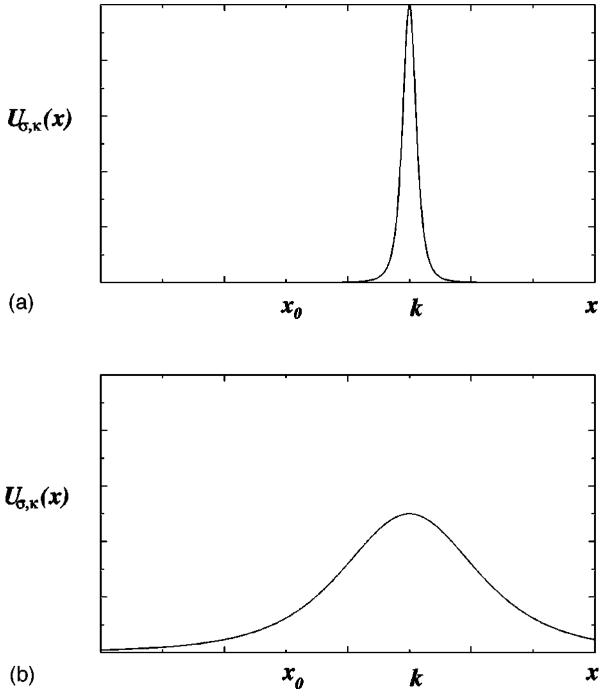

FIG. 3. The individual potential $U_{\sigma,k}(x)$ generated by the kth monomer at position $x=k$ in the polymer. (a) If the range of the potential is extremely short, or if σ has a very small value, then the contribution of $U_{\sigma,k}$ at position x_0 is completely negligible, provided k is far enough from x_0. (b) By increasing the range of the potential, or allowing σ to acquire larger and larger values, the potential ''spreads out'' having now an appreciable contribution at position x_0.

We now proceed to find $\mathcal{P}(V(x_1),V(x_2),\dots,V(x_r))$, namely, the r-point joint probability density function characterizing $V(x)$. Our aim is to show that in an appropriate, physically relevant limit, we can apply the central limit theorem (CLT) so that $V(x)$ becomes a Gaussian process. Note that in general the random variables $U^\alpha_{\sigma,k}(x)$ are not equally distributed and have different probability density functions. If we let $P_k(u)$ be the probability density function associated with the random variable $U^\alpha_{\sigma,k}(x)$, it follows from Eq. (2) that $P_k(u)$ and $P(a)$ are related by the following equation:

$$P_k(u)=\frac{1}{g^\alpha_{\sigma,k}(x)}P\left(\frac{u}{g^\alpha_{\sigma,k}(x)}\right), \tag{6}$$

where for simplicity we have defined $g^\alpha_{\sigma,k}(x)$ as

$$g^\alpha_{\sigma,k}(x)\equiv g^\alpha_\sigma(x-k). \tag{7}$$

Equation (6) shows that the random variables $U^\alpha_{\sigma,k}(x)$ are in general not equally distributed. Therefore, if we want to determine the probability distribution function $\mathcal{P}(V(x_1),V(x_2),\dots,V(x_r))$ characterizing the overall potential $V(x)$, we have to make use of the Lindeberg version of the CLT, which states that $\mathcal{P}(V(x_1),V(x_2),\dots,V(x_r))$ approaches a Gaussian distribution whenever the Lindeberg condition holds [7] (see the Appendix).

We now turn to the question of the ''appropriate limit'' in which the CLT can be applied to our problem. First, we should remark that the Lindeberg condition requires that the number of terms appreciably contributing to the sum in Eq. (4) be very large (formally, infinity). Note that having an infinite number of monomers in the polymer does not guarantee that the number of terms appreciably contributing to the overall potential is also infinite. This situation is depicted very schematically in Fig. 3, where we show the individual potential $U^\alpha_{\sigma,k}(x)$ generated by the kth monomer of the polymer, for two different values of the parameter σ. If σ is very small, we can make the contribution of $U^\alpha_{\sigma,k}(x)$ completely negligible at position x by taking k sufficiently far away from the position x. When we say ''completely negligible'' we mean that the $U^{\alpha}_{\sigma,k}(x)$ contribution is negligible with respect to that due to the nearest monomer to the x position. Under these circumstances, the sum in Eq. (4) will contain a reduced number of appreciably contributing terms, regardless of the polymer size. On the other hand, if σ is very large, then the potential $U^{\alpha}_{\sigma,k}(x)$ spreads out and its contribution is comparable to that coming from the nearest monomer to x. The above means that when the Q particle is very close to the polymer (very small σ), it tends to ''feel'' the monomer just below it, while if Q is far enough from the polymer, then a collective interaction between particle and polymer prevails, with more terms contributing significantly to Eq. (4) as σ increases.

From the above it is clear that, in order to apply the CLT, we must require both that the polymer has an infinite number of monomers and that σ has a large enough value. Under these circumstances, the Lindeberg condition stated for the sum in Eq. (4) is (see the Appendix)

$$\lim_{\sigma\to\infty}\frac{1}{J_0^2}\int_{-\infty}^{\infty}dy[g^{\alpha}_{\sigma}(y)]^2\int_{|a|>tJ_0/g^{\alpha}_{\sigma}(y)}a^2P(a)da=0, \quad (8)$$

for any real number $t>0$, where J_0^2 is given by

$$J_0^2\equiv\Delta^2\int_{-\infty}^{\infty}[g^{\alpha}_{\sigma}(y)]^2dy, \quad \Delta^2=\int_{-\infty}^{\infty}a^2P(a)da. \quad (9)$$

The Lindeberg condition is sufficient for the CLT to be valid. This condition ensures that J_0^2 [and therefore the random potential $V(x)$ given in Eq. (4)], is the sum of many contributions, each of them negligible compared with J_0^2 itself, and imposes three restrictions on the potentials $U^{\alpha}_{\sigma,k}(x)$ for which our theory applies, i.e., (1) $g^{\alpha}_{\sigma}(y)$ must be square integrable; (2) $g^{\alpha}_{\sigma}(y)/J_0\to 0$ as $\sigma\to\infty$; and (3) $P(a)$ must has a well defined second moment.

For this reason we have assumed that the potentials $U^{\alpha}_{\sigma,k}(x)$ are bounded or localized and that $P(a)$ is bounded in the sense that $P(a)=0$ if $|a|>R$. On the other hand, it is clear that in the rough potential case, the Lindeberg condition applies to the potential derivatives (i.e., the forces) or to the potential differences.

It is straightforward to show that the class of potentials $U^{\alpha}_{\sigma,k}(x)$ given in Eqs. (2) and (3) satisfy the Lindeberg condition if $\alpha>1/2$, and that if $-1/2\leqslant\alpha<1/2$ this condition is satisfied by the derivatives or differences of these potentials. Futhermore, in the Appendix we show that this condition is also satisfied by the wider class of ''screened potentials''

$$U^{\alpha\beta}_{\sigma,k}(x)=a_k g^{\alpha\beta}_{\sigma}(x-k), \quad (10)$$

$$g^{\alpha\beta}_{\sigma}(y)=\frac{\exp[-(\gamma/2)(y^2+\sigma^2)^{\beta}]}{(y^2+\sigma^2)^{\alpha/2}}, \quad (11)$$

as long as $0<\beta<1$, for every α. The factor $\exp[-(\gamma/2)(y^2+\sigma^2)^{\beta}]$ allows for the possibility that the potential between the polymer and the test particle is ''screened.'' In particular, if $\beta=1/2$, Eq. (11) transforms into a Yukawa potential, which in the literature is always referred to as a ''short range potential.'' However, from the above we can see that even the Yukawa potential has a long enough range to satisfy the Lindeberg condition.

If $\beta\geqslant 1$ the interaction becomes very short range, and under these circumstances the CLT does not hold. The reason is that in this case the individual potentials are of such extremely short range that for a given position x, we are not dealing with an infinite number of random variables in Eq. (4), but rather with just a few of them (those that are in the vicinity of the Q particle and therefore the only ones that appreciably contribute to the overall potential). The above considerations make it evident that the Lindeberg condition may not be satisfied either because the individual potentials do not decay to zero sufficiently rapidly with distance and consequently they are not bounded or localized (square integrable), or because the individual potentials decay extremely rapidly to zero with the distance (very short range or strictly finite range), which implies that at any position x only a small number of random variables contribute to the overall potential $V(x)$.

Once we have ensured that the Lindeberg condition holds for the class of potentials we are considering, it follows that the r-point joint probability distribution $\mathcal{P}(V(x_1),V(x_2),\ldots,V(x_r))$ will be Gaussian for very large σ. This is not to say that the resulting distribution cannot be Gaussian if the above condition does not hold, as the above is a sufficient but not a necessary condition. Indeed, if the Lindeberg condition is not satisfied, the random potential $V(x)$ may or may not be described by a Gaussian distribution. In what follows, we will focus our attention on potentials for which the Lindeberg condition is satisfied.

V. PROBABILITY DISTRIBUTION OF THE RANDOM POTENTIAL

In this section we present a simple formalism to compute $\mathcal{P}(V(x_1),V(x_2),\ldots,V(x_r))$ explicitly. In order to make our approach as simple as possible, we will focus on the two-point joint probability distribution function $\mathcal{P}(V(x_1),V(x_2))$. All the results presented here can easily be generalized to the case of r sites.

We start by assuming that the polymer is composed of a finite number of monomers, say n. The basic element in our analysis is that the potential $V(x)$ is a random variable consisting of the sum of the potentials $U^{\alpha}_{\sigma,k}(x)$ due to the n monomers in the polymer, and that these are independent random variables whose probability distributions are given by Eq. (6). The general scheme is to state a recurrence relation for the two-point joint probability distribution, labeled with the number n of monomers on the polymer, and to solve this recurrence relation in the limit $n\to\infty$ and for very large σ. As we have seen, the CLT ensures that the resultant two-point joint probability distribution $\mathcal{P}(V(x_1),V(x_2))$ will be a Gaussian distribution for which we need to determine the parameters characterizing it.

In what follows, it will be useful to distinguish between the interaction potential generated by a polymer with $n-1$ monomers and that generated by a polymer with n monomers. Thus, we will write as $V_{n-1}(x)$ and $V_n(x)$ the potential landscapes generated by the $(n-1)$-polymer and the n-polymer respectively. Suppose first that the polymer is made up of $n-1$ monomers whose charges are a_1, a_2, $\ldots$,

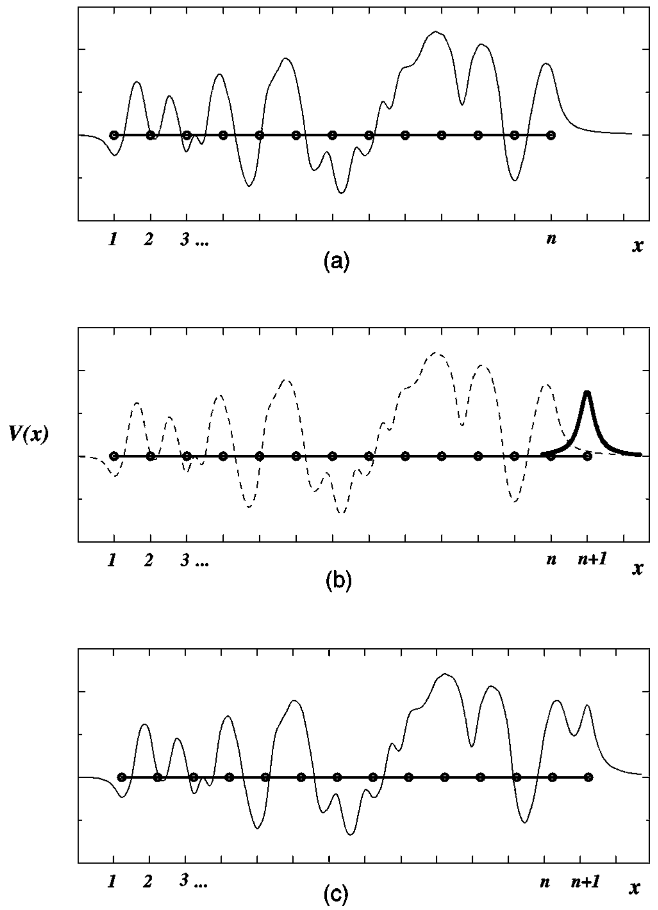

FIG. 4. (a) Potential landscape generated by a polymer made up of $n-1$ monomers. (b) Another monomer is added at position $x=n$. This new monomer contributes to the overall potential with the term $U_{\sigma,n}(x)$ (solid line). (c) The resulting potential landscape after the incorporation of the nth monomer contribution. As can be seen, the curve is different from the one depicted in (a); hence, the probability distribution also changes.

a_{n-1}. These charges generate the interaction potential $V_{n-1}(x)=\Sigma_{k=1}^{n-1}a_k g_{\sigma,k}^{\alpha}(x)$ which is characterized by the two-point probability distribution $\mathcal{P}_{n-1}(V_{n-1}(x_1),V_{n-1}(x_2))$. When we add another monomer with charge a_n to the polymer end (see Fig. 4), it contributes to the overall potential with the term $U_{\sigma,n}^{\alpha}(x)=a_n g_{\sigma,n}^{\alpha}(x)$, so that now the interaction potential is given by $V_n(x)=V_{n-1}(x)+a_n g_{\sigma,n}^{\alpha}(x)=\Sigma_{k=1}^{n}a_k g_{\sigma,k}^{\alpha}(x)$. The new term changes the potential landscape, transforming its probability distribution into $\mathcal{P}_n(V_n(x_1),V_n(x_2))$. Now, if $V_n(x)$ takes the specific values $V_n(x_1)$ and $V_n(x_2)$ at x_1 and x_2, respectively, then $V_{n-1}(x)$ *should have* the values $V_{n-1}(x_1)=V_n(x_1)-a_n g_{\sigma,n}^{\alpha}(x_1)$ and $V_{n-1}(x_2)=V_n(x_2)-a_n g_{\sigma,n}^{\alpha}(x_2)$ at the corresponding positions, and the probability for this to happen is

$$\mathcal{P}_{n-1}(V_{n-1}(x_1),V_{n-1}(x_2)) \\ =\mathcal{P}_{n-1}[V_n(x_1)-a_n g_{\sigma,n}^{\alpha}(x_1),V_n(x_2)-a_n g_{\sigma,n}^{\alpha}(x_2)]. \tag{12}$$

We can now obtain $\mathcal{P}_n(V_n(x_1),V_n(x_2))$ in terms of the last equation, multiplying it by $P(a_n)$ and then summing up over all the possible values of a_n leading to the values $V_n(x_1)$ and $V_n(x_2)$ at positions x_1 and x_2. Taking into account that $P(a)$ is the probability distribution of all the a_k's, the relationship between $\mathcal{P}_{n-1}$ and $\mathcal{P}_n$ is given by the following integral relation:

$$\mathcal{P}_n(V_n(x_1),V_n(x_2))=\int_{-\infty}^{\infty}\mathcal{P}_{n-1}(V_n(x_1)-a g_{\sigma,n}^{\alpha}(x_1),V_n(x_2) \\ -a g_{\sigma,n}^{\alpha}(x_2))P(a)da, \tag{13}$$

This last equation can be handled better in Fourier space, since there the convolution transforms into a simple product. Thus, if we call ξ_1 and ξ_2 the Fourier conjugate variables of $V_n(x_1)$ and $V_n(x_2)$, respectively, by Fourier transforming Eq. (13) we arrive at the equivalent expression

$$\hat{\mathcal{P}}_n(\xi_1,\xi_2)=\hat{\mathcal{P}}_{n-1}(\xi_1,\xi_2)\hat{P}(g_{\sigma,n}^{\alpha}(x_1)\xi_1+g_{\sigma,n}^{\alpha}(x_2)\xi_2). \tag{14}$$

In order to solve this recurrence equation we need to determine $\mathcal{P}_1(V_1(x_1),V_1(x_2))$, namely, the two-point joint probability distribution in the case when there is just one monomer in the ''polymer.'' To do this, we should note that, in such a case, with just one particle placed at $x=1$, the overall potential $V(x)$ is simply given by

$$V(x)=V_1(x)=a_1 g_{\sigma,1}^{\alpha}(x). \tag{15}$$

Thus the potential will have the value $V_1(x_1)$ at $x=x_1$ if and only if the variable a_1 has the value

$$a_1=\frac{V_1(x_1)}{g_{\sigma,1}^{\alpha}(x_1)}. \tag{16}$$

The probability for this to happen is given by Eq. (6), with $k=1$:

$$P_1(V_1(x_1))=\frac{1}{g_{\sigma,1}^{\alpha}(x_1)}P\left(\frac{V_1(x_1)}{g_{\sigma,1}^{\alpha}(x_1)}\right). \tag{17}$$

Once $V(x)$ has acquired the value $V_1(x_1)$ at $x=x_1$, its value $V_1(x_2)$ at another position $x=x_2$ is constrained to be

$$V_1(x_2)=a_1 g_{\sigma,1}^{\alpha}(x_2)=\frac{V_1(x_1)}{g_{\sigma,1}^{\alpha}(x_1)}g_{\sigma,1}^{\alpha}(x_2), \tag{18}$$

and the conditional probability $P(V(x_1)|V(x_2))$ for this to happen is simply

$$P(V(x_1)|V(x_2))=\delta\left(V_1(x_2)-\frac{V_1(x_1)}{g_{\sigma,1}^{\alpha}(x_1)}g_{\sigma,1}^{\alpha}(x_2)\right)$$

where $\delta(\cdot)$ is the Dirac delta function. Therefore, the probability distribution $\mathcal{P}_1(V_1(x_1),V_1(x_2))$ is

$$\mathcal{P}_1(V_1(x_1),V_1(x_2)) = \frac{1}{g^{\alpha}_{\sigma,1}(x_1)} P\left(\frac{V_1(x_1)}{g^{\alpha}_{\sigma,1}(x_1)}\right) \times \delta\left(V_1(x_2) - \frac{V_1(x_1)}{g^{\alpha}_{\sigma,1}(x_1)} g^{\alpha}_{\sigma,1}(x_2)\right) \tag{19}$$

By Fourier transforming the above equation we have

$$\hat{\mathcal{P}}_1(\xi_1,\xi_2) = \hat{P}(g^{\alpha}_{\sigma,1}(x_1)\xi_1 + g^{\alpha}_{\sigma,1}(x_2)\xi_2), \tag{20}$$

This is the initial condition for Eq. (14), which then becomes

$$\hat{\mathcal{P}}_n(\xi_1,\xi_2) = \prod_{k=1}^{n} \hat{P}(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2). \tag{21}$$

For the sake of symmetry, we label the particles from $-m$ to m, and will eventually take the limit $m\to\infty$, indicating that all the particles in the polymer are being considered. The required solution to the recurrence relation Eq. (14) is then

$$\hat{\mathcal{P}}_m(\xi_1,\xi_2) = \prod_{k=-m}^{m} \hat{P}(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2). \tag{22}$$

Now, we are assuming that the functions $g^{\alpha}_{\sigma,k}(x)$ vanish as σ grows [see Eq. (3) or Eq. (11)]. Therefore, if σ is very large, the argument $\lambda = g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2$ of $\hat{P}(\lambda)$ in Eq. (22) is very small, and we can expand this function in powers of its argument, keeping up to second order terms:

$$\begin{aligned}\hat{P}(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2) &\simeq 1 - i\mu(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2) \\ &\quad - \tfrac{1}{2}\Delta^2(g^{\alpha}_{\alpha,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2)^2 + \cdots,\end{aligned} \tag{23}$$

where, of course, μ and Δ^2 are the first and second moments of the probability distribution $P(a)$, respectively. In view of the condition (a) imposed on $P(a)$ (see Sec. II), $\mu=0$, and consequently Eq. (22) can be written as

$$\hat{\mathcal{P}}_m(\xi_1,\xi_2) \simeq \prod_{k=-m}^{m} [1 - \tfrac{1}{2}\Delta^2(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2)^2]. \tag{24}$$

At this point it is worth while to emphasize that we are working under the assumption that the hypothesis for the central limit theorem holds. Thus, though we only have a second order expression in Eq. (24), this theorem ensures that the term $1 - \frac{1}{2}\Delta^2(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2)^2$ necessarily comes from the expansion of $\exp[-\frac{1}{2}\Delta^2(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2)^2]$. Equation (24) can then be written as

$$\begin{aligned}\hat{\mathcal{P}}_m(\xi_1,\xi_2) &\simeq \prod_{k=-m}^{m} \exp[-\tfrac{1}{2}\Delta^2(g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2)^2] \\ &= \exp\left(-\frac{1}{2}\Delta^2 \sum_{k=-m}^{m} [g^{\alpha}_{\sigma,k}(x_1)\xi_1 + g^{\alpha}_{\sigma,k}(x_2)\xi_2]^2\right).\end{aligned}$$

The above is clearly the characteristic function corresponding to a Gaussian distribution, as expected. In the limit $m\to\infty$ and for very large σ, the sums in the last expression can be substituted by integrals, and this characteristic function acquires the final form

$$\hat{\mathcal{P}}(\xi_1,\xi_2) = \exp[-\tfrac{1}{2}(J_0^2\xi_1^2 + J_0^2\xi_2^2 + J_{1,2}^2\xi_1\xi_2)], \tag{25}$$

where J_0^2 and $J_{1,2}^2$ are defined as

$$J_0^2 = \Delta^2 \int_{-\infty}^{\infty} [g^{\alpha}_{\sigma}(z)]^2 dz, \tag{26}$$

$$J_{1,2}^2 = \Delta^2 \int_{-\infty}^{\infty} g^{\alpha}_{\sigma}(z-x_1) g^{\alpha}_{\sigma}(z-x_2) dz. \tag{27}$$

By taking the inverse Fourier transform of Eq. (25) we obtain the desired two-point probability distribution $\mathcal{P}(V(x_1),V(x_2))$, which is then

$$\begin{aligned}\mathcal{P}(V(x_1),V(x_2)) &= \frac{1}{\pi\sqrt{4J_0^4 - J_{1,2}^4}} \exp\left(-\frac{1}{4} \times \frac{J_0^2[V(x_1)]^2 + J_0^2[V(x_2)]^2 - J_{1,2}^2 V(x_1)V(x_2)}{4J_0^4 - J_{1,2}^4}\right).\end{aligned} \tag{28}$$

The above result shows explicitly that the process we are dealing with is a stationary Gaussian process characterized by J_0^2 and $J_{1,2}^2$. Indeed, $J_{1,2}^2$ is the correlation function of the random potential, which is a function only of the separation distance $|x_1 - x_2|$. The long distance correlation of the potential can be obtained by Fourier transforming Eq. (27) and then examining its small wave number behavior. Since $J_{1,2}^2$ is a convolution, its Fourier transform $\widehat{J_{1,2}^2}(\lambda)$ is straightforward:

$$\widehat{J_{1,2}^2}(\lambda) = \Delta^2[\hat{g}^{\alpha}_{\sigma}(\lambda)]^2. \tag{29}$$

The detailed behavior will depend on the nature of the interaction, but in general terms, we can say that for the bounded and localized potential cases $\widehat{J_{1,2}^2}(\lambda)\to$const as $\lambda\to 0$; otherwise, $\widehat{J_{1,2}^2}(\lambda) \sim \lambda^{-\nu}$ for small λ, where the exponent ν is a measure of the "roughness" of the potential. We have mentioned above that in the rough potential case it is convenient to look at the potential differences or the potential derivatives instead of the potential itself. It can be shown that for the potentials $U^{\alpha}_{\sigma,n}(x)$ that we have referred to in

Eqs. (2) and (3) $\nu=1-2\alpha$ and hence the long distance behavior of the overall potential $V(x)$ is given by

$$\langle [V(x_1)-V(x_2)]^2\rangle \to \begin{cases} f(\alpha)|x_1-x_2|^{1-2\alpha} & \alpha\leqslant 1/2 \\ \text{const} & \alpha>1/2 \end{cases}$$

as $|x_1-x_2|\to\infty$, where $f(\alpha)$ is a function that vanishes at $\alpha=0$.

VI. DYNAMICS

We now illustrate how the results of the preceding sections can be used to determine some statistical properties related to the dynamics of a test particle Q moving through a finite polymer made up of $n=2m+1$ monomers ($m\to\infty$). In order to do this, we suppose that there is an external driving force F acting on the particle whose only purpose is to make the particle go through the polymer, ''exploring'' the potential. We will restrict our analysis to the *bounded force case*. This is analogous to the bounded potential case which we have referred to in Sec. III, but with forces instead of potentials. Under such circumstances, we can always choose the value of the force F sufficiently large to avoid the particle being trapped in one of the potential minima.

In principle, the force F can be time dependent, but in our analysis we will use a steplike force given by

$$F=\begin{cases} F_0, & x\in[-m,m] \\ 0, & x\notin[-m,m], \end{cases} \tag{30}$$

where F_0 is a constant. The dynamics of the particle in the overdamped regime [8] is given by

$$\gamma v=-\frac{\partial V(x)}{\partial x}+F, \tag{31}$$

or taking into account Eq. (4) we have

$$\gamma v=-\frac{\partial}{\partial x}\left(\sum_{k=-m}^{m} a_k g_\sigma^\alpha(x-k)\right)+F. \tag{32}$$

In Fig. 5(a) we show the velocity as a function of the position of the particle along the polymer, where we have used the potential of Fig. 2(c). On the other hand, Fig. 5(b) shows the power spectrum of the above velocity, numerically obtained by means of Lomb's method [9]. The spectrum shows mild frequency selection as well as a hump in the average behavior, which, as we will see, is related to statistical properties of the potential disorder [10]. For this, we Fourier transform Eq. (32), which leads to

$$\hat{v}(\lambda)=i\lambda\hat{g}_\sigma^\alpha(\lambda)\sum_{k=-m}^{m} a_k e^{-i\lambda k}-\frac{2F_0}{\lambda}\sin(m\lambda), \tag{33}$$

where, for simplicity, we have taken $\gamma=1$. The second term on the right hand side of Eq. (33) is just the Fourier transform of the constant force F given in Eq. (30). Since such a term does not carry information about the spatial structure, we can drop it from the analysis, retaining only the first term. In doing so, we keep in the power spectrum just the information concerning the potential structure. Nevertheless, we should keep in mind that the external force F must be present in Eq. (32) in order to drive the system.

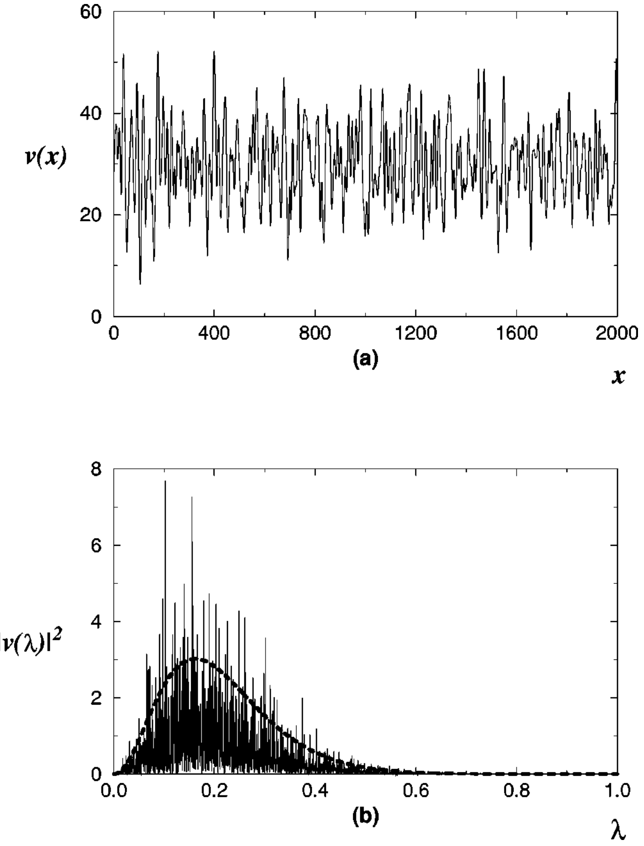

FIG. 5. (a) Single realization of the velocity of the particle along the polymer as a function of the position, corresponding to the potential shown in Fig. 2(c). (b) Fourier spectrum of the above velocity. The dashed bold curve is the corresponding averaged power spectrum $\langle|\hat{v}(\lambda)|^2\rangle$ given in Eq. (37).

With the above considerations, the contribution to the power spectrum of the velocity relevant for our purposes is given by

$$|\hat{v}(\lambda)|^2=\lambda^2[\hat{g}_\sigma^\alpha(\lambda)]^2\left|\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right|^2. \tag{34}$$

If we take the average of this equation over the ensemble of all the possible realizations of the charges a_k, we obtain

$$\langle|\hat{v}(\lambda)|^2\rangle=n\lambda^2\Delta^2[\hat{g}_\sigma^\alpha(\lambda)]^2 \tag{35}$$

see the Appendix, or taking into account Eq. (29) we get

$$\langle|\hat{v}(\lambda)|^2\rangle=n\lambda^2\widehat{J_{1,2}^2}(\lambda). \tag{36}$$

with $n=m+1$ the total number of monomers in the polymer. The above equation shows that the statistical features of the velocity exhibited in the power spectrum are a direct consequence of the potential correlations.

For the class of bounded potentials $U_{\sigma,n}^\alpha(x)$ given in Eqs. (2) and (3) with $\alpha>1$, the average power spectrum of the velocity is as follows:

$$\langle|\hat{v}(\lambda)|^2\rangle = \frac{n8\pi\Delta^2}{[\Gamma(\alpha/2)]^2 2^{\alpha}\sigma^{2\alpha}}[(\sigma\lambda)^{(\alpha+1)/2}K_{(\alpha-1)/2}(\sigma\lambda)]^2, \tag{37}$$

where $K_\beta(z)$ is the Bessel function of the second kind of order β. In Fig. 5(b) we show that the graph of Eq. (34) acts as an envelope of the power spectrum. Also note from the above equation that $\langle|\hat{v}(\lambda)|^2\rangle$ is actually a function of the scaled variable $z=\sigma\lambda$. Multipling the wave number λ by σ is equivalent, in real space, to taking the transverse distance σ as the unit of length (instead of the separation distance between the monomers).

Since the potentials we are considering are Gaussian processes, following Rice [4] we can write an explicit relation between statistical features of the potential and dynamical properties, namely,

$$\overline{l} = 4\pi\left(\frac{\int_0^\infty \langle|\hat{v}(\lambda)|^2\rangle d\lambda}{\int_0^\infty \lambda^2\langle|\hat{v}(\lambda)|^2\rangle d\lambda}\right)^{1/2}. \tag{38}$$

where $\overline{l}$ is the mean distance between consecutive minima in the velocity of the particle along the polymer. Combining the last two equations we have that the mean distance $\overline{l}_\alpha$ between consecutive minima in the velocity as

$$\overline{l}_\alpha = 4\pi\sigma h(\alpha), \tag{39}$$

where

$$h(\alpha) = \left(\frac{\int_0^\infty [z^{(\alpha+1)/2}K_{(\alpha-1)/2}(z)]^2 dz}{\int_0^\infty [z^{(\alpha+3)/2}K_{(\alpha-1)/2}(z)]^2 dz}\right)^{1/2} \tag{40}$$

is a function of α only. In Fig. 6 we show the graph of $\overline{l}_\alpha/\sigma$ as a function of α, obtained by numerically computing $h(\alpha)$ from Eq. (40). By performing a least squares fit on this graph, the functional relationship between α and $\overline{l}_\alpha/\sigma$ turns out to be

$$\overline{l}_\alpha/\sigma \cong 5.13\alpha^{-1/2} \tag{41}$$

Note that Eq. (39) states that $\overline{l}_a$ is a *scale invariant quantity*, since for a given type of interaction (fixed α), $\overline{l}_\alpha/\sigma$ is constant, provided σ is large enough for the Gaussian approximation to hold.

The above considerations can be visualized in Fig. 7, where we show the velocity of the particle along the polymer for different values of σ and $\alpha=4$. As can be seen in the figure, if σ is large enough, the statistical structure of the velocity does not change as σ increases, once we have rescaled the graphs in units of σ. In Fig. 8 the velocity of the particle along the polymer is depicted again, but now for successively increasing values of α and constant σ. We can see that Eq. (41) is actually giving the mean distance between consecutive minima in the velocity by counting the number of relevant minima occurring in each graph of Fig. 8, and dividing the corresponding interval on the X axis by the respective number of minima. For example, in Fig. 8(b) ($\alpha=20$) the number of relevant minima is $N=31$, and the interval on the X axis is $I_x=50$ (in σ units). Consequently, the observed mean distance in this figure is $N/I_x=50/31\approx 1.613$, in good agreement with the $\overline{l}_{20}/\sigma=1.622$ value predicted by Eq. (41).

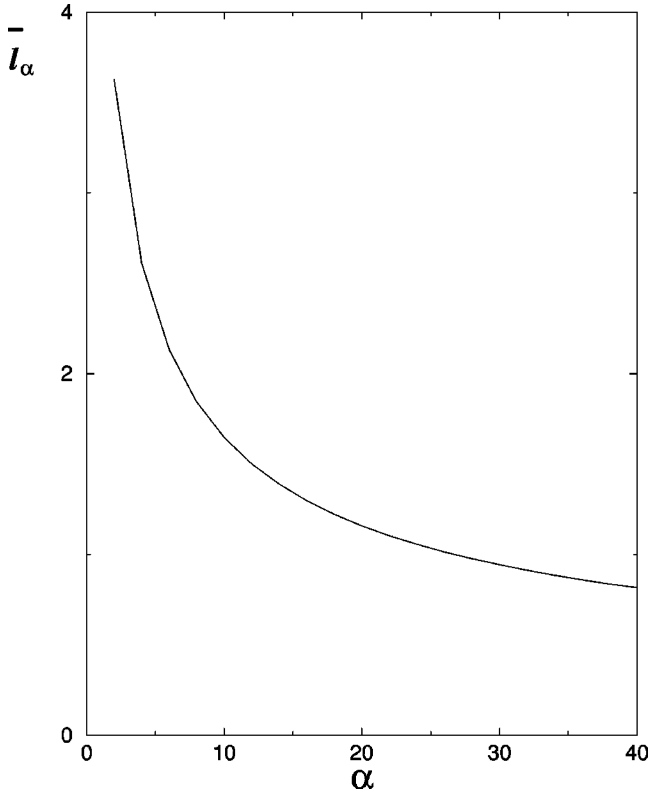

FIG. 6. Mean distance $\overline{l}_\alpha$ (in σ units) between consecutive minima in the velocity as a function of α.

VII. CONCLUDING REMARKS

Throughout this work we have seen that a random distribution of charges along a polymer generates a random correlated potential with a Gaussian distribution, even though the charges on the polymer are not correlated. This result is quite general since, as we have seen, in order to obtain it very few restrictions on the system were imposed: for the electrostatic potentials most commonly occurring in nature (which are square integrable), it is sufficient for the distribution $P(a)$ to have a well defined second moment.

In view of the results presented here, a note of caution is relevant when assigning correlations in the modeling of disorder. We have shown that for randomly charged point particles, the CLT restricts the resulting potentials to be correlated Gaussian processes; for example, a δ-correlated potential landscape $V(x)$, i.e., one whose correlation function is given by $\langle V(x_1)V(x_2)\rangle = \delta(x_1-x_2)$ is ruled out.

Although the electrostatic potentials most commonly occurring between point particles actually satisfy the Lindeberg condition, and therefore yield Gaussian distributions, there

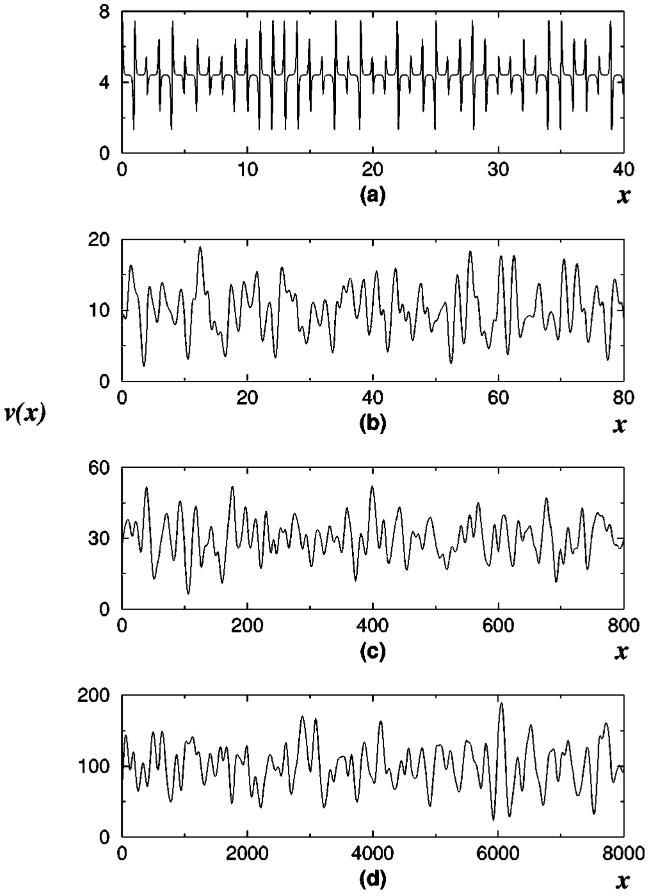

FIG. 7. Velocity of the particle along the polymer for different values of σ and $\alpha=4$. (a) $\sigma=0.1$. (b) $\sigma=1$. (c) $\sigma=10$. (d) $\sigma=100$. Note that the statistical structure of the velocity no longer changes for large values of σ. The graphs were obtained by using the same probability distribution of charges as in Fig. 2.

are some interaction potentials for which the Lindeberg condition does not hold. If the individual potentials generated by each one of the monomers are of extremely short range, or if, on the contrary, they do not decrease sufficiently rapidly with distance, then the Lindeberg condition is no longer valid and consequently the resultant potential will not necessarily be a Gaussian process. In Table I we show parameter ranges for which the ''screened'' interaction potential given by Eq. (11) satisfies or not the Lindeberg condition. We can consider these potentials (or a linear superposition of them) as the most general type of electrostatic potentials occurring between point particles.

We should emphasize that, in the rough potential case, the Gaussian probability distribution corresponds not to the potential itself, but rather to the potential differences or potential derivatives, which are the physically relevant quantities.

We have also been able to establish links between statistical properties of the potential and dynamical features of the test particle interacting with the polymer. In the Gaussian approximation, the power spectrum of the particle's velocity is scale invariant and directly related to the long range correlations of the potential; moreover, the mean distance $\overline{l}_\alpha$ between consecutive minima in the velocity of the particle is a scale invariant quantity over all length scales for which a collective interaction between the particle and the monomers

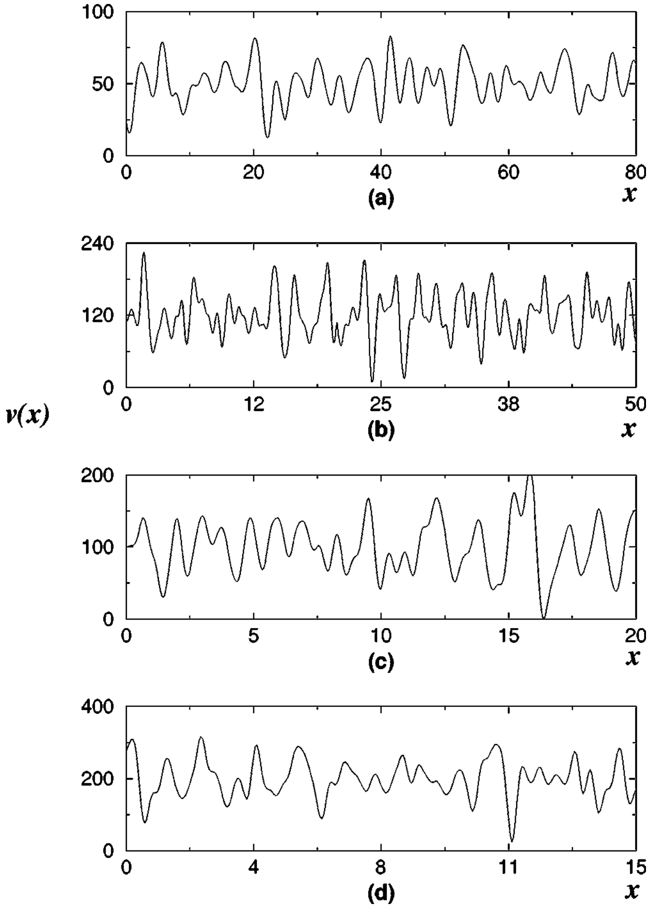

FIG. 8. Velocity of the particle through the polymer for different values of α and $\sigma=100$, plotted as a function of the x position measured in σ units (x/σ). The charge probability distribution is the same as in Fig. 2. (a) In this case $\alpha=2$. The number of relevant minima is $N=22$ and the interval length is $I_x=80$. So the mean distance between minima in the interval shown is $\overline{l}_2=80/22=3.6363$. (b) $\alpha=10$, $N=31$, and $I_x=50$; therefore $\overline{l}_{10}=50/31=1.6129$. (c) $\alpha=20$ and $\overline{l}_{20}=20/17=1.1764$. (d) $\alpha=40$ and $\overline{l}_{40}=15/18=0.8333$.

TABLE I. Parameter ranges for which the ''screened'' interaction potential given by Eq. (11) satisfies or does not satisfy the Lindeberg condition.

$$U(y)=\frac{1}{[y^2+\sigma^2]^{\alpha/2}}\exp\left(-\frac{\gamma}{2}(y^2+\sigma^2)^\beta\right)$$

Lindeberg condition holds	
$0<\beta<1$ and any α	Bounded potential
$\beta=0$ and $\alpha>1$	Bounded potential
$\beta=0$ and $1/2<\alpha\leqslant 1$	Localized potential
$\beta=0$ and $-1/2<\alpha\leqslant 1/2$	Rough potential
Lindeberg condition does not hold	
$\beta>1$ and any α	Very short range potential
$\beta=0$ and $\alpha\leqslant -1/2$	Nondifferentiable potential

prevails (i.e., for sufficiently large values of σ).

Finally, we believe that it is important to determine the distribution function of the distance between consecutive minima in the velocity of the particle (or in the potential landscape), in order to calculate some quantities related to the transport along a random potential, such as relaxation or passage times. Also, the analysis that we have done should be extended to more than one dimension, and to potentials with correlated charge distributions. The latter may produce processes beyond the correlated Gaussian ones we have dealt with here. Work in this direction is in progress.

ACKNOWLEDGMENTS

We would like to acknowledge support from DGAPA-UNAM under Project No. IN116198. M. Aldana acknowledges DGAPA-UNAM for financial support. We are also grateful to Dr. Germinal Cocho and Dr. Francois Leyvraz for their useful comments and discussions.

APPENDIX

1. Lindeberg Condition

The Lindeberg condition can be stated as follows [7]: *Let* $\xi_1,\xi_2,\ldots,\xi_m$ *be independent random variables whose probability distributions are* $P_1,P_2,\ldots,P_m$, *respectively. Suppose that these distributions are such that the mean value* $\boldsymbol{E}(\xi_k)=0$, $\mathrm{Var}(\xi_k)=\delta_k^2$, *and let* s_m^2 *be*

$$s_m^2=\delta_1^2+\delta_2^2+\cdots+\delta_m^2. \tag{A1}$$

If for every $t>0$ *the following equality holds*:

$$\lim_{m\to\infty} s_m^{-2}\sum_{k=1}^{m}\int_{|y|>ts_m} y^2P_k(y)\,dy=0, \tag{A2}$$

then the probability distribution of the normalized sum

$$\chi_m=(\xi_1+\xi_2+\cdots+\xi_m)/s_m$$

will tend to a Gaussian distribution with zero mean and unitary variance as $m\to\infty$.

In the problem we are considering, the overall potential $V(x)$ is given by

$$V(x)=\sum_{k=-m}^{m}U_{\sigma,k}^{\alpha}(x)$$

[see Eq. (4)], where each $U_{\sigma,k}^{\alpha}(x)$ is a random variable whose probability density $P_k(u)$ is as in Eq. (6). For a fixed position x, the variance $\delta_k^2(x)$ of the individual random potential $U_{\sigma,k}^{\alpha}(x)$ is given by

$$\begin{aligned}\delta_k^2(x)&=\int_{-\infty}^{\infty}[U_{\sigma,k}^{\alpha}(x)]^2P_k(U_{\sigma,k}^{\alpha}(x))\,dU_{\sigma,k}^{\alpha}(x)\\&=\int_{-\infty}^{\infty}[ag_{\sigma,k}^{\alpha}(x)]^2\frac{1}{g_{\sigma,k}^{\alpha}(x)}P\left(\frac{U_{\sigma,k}^{\alpha}(x)}{g_{\sigma,k}^{\alpha}(x)}\right)d[ag_{\sigma,k}^{\alpha}(x)]\\&=[g_{\sigma,k}^{\alpha}(x)]^2\int_{-\infty}^{\infty}a^2P(a)\,da=\Delta^2[g_{\sigma,k}^{\alpha}(x)]^2,\end{aligned}$$

where as before Δ^2 is the second moment of $P(a)$. Therefore, the s_m^2 variable of Eq. (A1) is

$$s_m^2=\Delta^2\sum_{k=-m}^{m}[g_{\sigma,k}^{\alpha}(x)]^2.$$

It is clear that in the limit $m\to\infty$ and for large values of σ, the sum in the above expression can be approximated by an integral, giving

$$J_0^2\equiv\lim_{m\to\infty}s_m^2=\Delta^2\int_{-\infty}^{\infty}[g_{\sigma}^{\alpha}(y)]^2dy. \tag{A3}$$

Consequently, the Lindeberg condition, when applied to the random potential $V(x)$, transforms into

$$\lim_{\sigma\to\infty}\lim_{m\to\infty}\frac{1}{s_m^2}\sum_{k=-m}^{m}\int_{|u|>ts_m}u^2P_k(u)\,du=0,$$

where we have also taken the limit $\sigma\to\infty$ attending to the considerations made in Sec. IV. Taking into account Eq. (6) and Eq. (A3), and replacing the sum by an integral, the above equation is equivalent to

$$\lim_{\sigma\to\infty}\frac{1}{J_0^2}\int_{-\infty}^{\infty}dy[g_{\sigma}^{\alpha}(y)]^2\int_{|a|>tJ_0/g_{\sigma}^{\alpha}(y)}a^2P(a)\,da=0,$$

which is Eq. (8).

2. Well behaved potentials

According to Sec. IV, to prove that some kind of potentials $U_\sigma(x)=ag_\sigma(x)$ satisfy the Lindeberg condition, we must show that

$$\lim_{\sigma\to\infty}I(\sigma)=0, \tag{A4}$$

where $I(\sigma)$ is defined as

$$I(\sigma)=J_0^{-2}\int_{-\infty}^{\infty}dyg_{\sigma}(y))^2\int_{|a|>tJ_0/g_{\sigma}(y)}a^2P(a)\,da. \tag{A5}$$

Suppose that $g_\sigma(x)$ is given as in Eq. (11):

$$g_\sigma(x)=a\frac{\exp[-(\gamma/2)(x^2+\sigma^2)^{\beta}]}{(x^2+\sigma^2)^{\alpha/2}}$$

(we have suppressed the superscripts α and β in order to simplify the notation). Then $J_0^2=\Delta^2\int_{-\infty}^{\infty}[g_\sigma(y)]^2dy$ is given by

$$J_0^2=\Delta^2\sigma^{1-2\alpha}\int_{-\infty}^{\infty}\frac{\exp[-\gamma\sigma^{2\beta}(1+t^2)^{\beta}]}{(1+t^2)^{\alpha}}dt. \tag{A6}$$

If σ is very large, then the exponential factor in the above integral goes to zero very quickly, and the main contribution to the integral comes from a very narrow interval around the origin. Therefore, for large values of σ we can approximate J_0^2 as

$$J_0^2 \approx \Delta^2 \sigma^{1-2\alpha} \int_{-\varepsilon}^{\varepsilon} \frac{\exp[-\gamma\sigma^{2\beta}(1+t^2)^\beta]}{(1+t^2)^\alpha} dt,$$

where $\varepsilon \ll 1$. We can further expand $(1+t^2)^\beta$ in powers of t, retaining up to the second order terms, and in view of the small value of ε we can simply take $(1+t^2)^\alpha = 1$, which lead us to

$$J_0^2 \approx \Delta^2 \sigma^{1-2\alpha} e^{\gamma\sigma^{2\beta}} \int_{-\varepsilon}^{\varepsilon} e^{-\gamma\beta\sigma^{2\beta}t^2} dt$$

$$\approx \Delta^2 \sqrt{\frac{\pi}{\gamma\beta}} \sigma^{1-\beta-2\alpha} e^{-\gamma\sigma^{2\beta}}. \tag{A7}$$

Now, we need to show that $g_\sigma(x)/J_0 \to 0$ as $\sigma \to \infty$. To do this, we note that for very large σ the square of the function $g_\sigma(x)$ can be written as

$$[g_\sigma(x)]^2 = \sigma^{-2\alpha} \frac{\exp[-\gamma\sigma^{2\beta}(1+\beta(x^2/\sigma^2)+\cdots)]}{1+\alpha(x^2/\sigma^2)+\cdots}.$$

Therefore, $[g_\sigma(x)/J_0]^2$ is

$$\left(\frac{g_\sigma(x)}{J_0}\right)^2 \approx \frac{1}{\Delta^2}\sqrt{\frac{\gamma\beta}{\pi}}\sigma^{2\alpha+\beta-1}e^{\gamma\sigma^{2\beta}}\sigma^{2\alpha}$$

$$\times \frac{\exp[-\gamma\sigma^{2\beta}(1+\beta(x^2/\sigma^2)+\cdots)]}{1+\alpha(x^2/\sigma^2)+\cdots}$$

$$\approx \frac{1}{\Delta^2}\sqrt{\frac{\gamma\beta}{\pi}}\sigma^{\beta-1}\frac{\exp[-\gamma\sigma^{2\beta}(\beta(x^2/\sigma^2)+\cdots)]}{1+\alpha(x^2/\sigma^2)+\cdots},$$

and it is easy to see that the above expression vanishes as $\sigma \to \infty$. Thus, we can make $J_0/g_\sigma(x)$ bigger than any real number just by choosing σ large enough.

On the other hand, we are assuming that $P(a)$ has a well defined second moment and therefore $P(a)$ must decay to zero faster than $1/|a|^3$ as $|a| \to \infty$. Hence, we can suppose, without loss of generality, that $P(a) \sim 1/|a|^{3+\varepsilon}$ as $|a| \to \infty$, for some $\varepsilon > 0$. The above, of course, represents an upper bound for the asymptotic behavior of $P(a)$.

As we have seen, if σ is very large, then $J_0/g_\sigma(y)$ is also very large and therefore we can use the above asymptotic behavior of $P(a)$ to compute $I(\sigma)$, which leads to

$$I(\sigma) \approx J_0^{-2} \int_{-\infty}^{\infty} dy [g_\sigma^\alpha(y)]^2 \int_{t|a|>tJ_0/g_\sigma(y)} a^2 \frac{1}{|a|^{3+\varepsilon}} da$$

$$= \frac{2t^{-\varepsilon}}{\varepsilon} \int_{-\infty}^{\infty} \left(\frac{g_\sigma(y)}{J_0}\right)^{2+\varepsilon} dy$$

$$= \frac{2t^{-\varepsilon}}{\varepsilon} J_0^{-(2+\varepsilon)} \int_{-\infty}^{\infty} [g_\sigma(y)]^{2+\varepsilon} dy$$

$$= \frac{2t^{-\varepsilon}}{\varepsilon} J_0^{-(2+\varepsilon)} \sigma^{1-2\alpha'} \int_{-\infty}^{\infty} \frac{\exp[-\gamma'\sigma^{2\beta}(1+y^2)^\beta]}{(1+y^2)^{\alpha'}} dy,$$

where $\alpha' = \alpha(1+\varepsilon/2)$ and $\gamma' = \gamma(1+\varepsilon/2)$. But the last integral is exactly the same as the one in Eq. (A6) with the parameters α and γ replaced by α' and γ', respectively. The value of this integral for very large σ is given in Eq. (A7) (with α' and γ' instead of α and γ). Therefore, the asymptotic dependence of $I(\sigma)$ on σ is given by

$$I(\sigma) \approx \frac{2t^{-\varepsilon}}{\Delta^{2+\varepsilon}\varepsilon\sqrt{1+\varepsilon/2}} \left(\frac{\gamma\beta}{\pi}\right)^{\varepsilon/4} \sigma^{(\beta-1)\varepsilon/2},$$

which evidently vanishes as $\sigma \to \infty$, provided $\beta < 1$. Thus, if $0 \leq \beta < 1$ the Lindeberg condition is satisfied, as we want to show.

3. Fourier transforms

In this section we explain how to obtain Eq. (35). We start by Fourier transforming Eq. (32), which leads to

$$\hat{v}(\lambda) = i\lambda \hat{g}_\sigma^\alpha(\lambda) \sum_{k=-m}^{m} a_k e^{-i\lambda k} + \frac{2F_0}{\lambda}\sin(m\lambda). \tag{A8}$$

The power spectrum of the velocity is given by $|\hat{v}(\lambda)|^2$. So by multiplying Eq. (A8) by its complex conjugate we obtain

$$|\hat{v}(\lambda)|^2 = \lambda^2 |\hat{g}_\sigma^\alpha(\lambda)|^2 \left|\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right|^2 + \frac{4F_0^2}{\lambda^2}\sin^2(m\lambda)$$

$$+ 4F_0 \hat{g}_\sigma^\alpha(\lambda)\sin(m\lambda)\mathrm{Re}\left(\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right), \tag{A9}$$

where we have used the fact that $\hat{g}_\sigma^\alpha(\lambda)$ is a real function. By averaging Eq. (A9) in the ensemble of all possible realizations of the a_k charges, using the probability density function $P(a)$, we obtain

$$\langle|\hat{v}(\lambda)|^2\rangle = \lambda^2 |\hat{g}_\sigma^\alpha(\lambda)|^2 \left\langle \left|\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right|^2 \right\rangle + \frac{4F_0^2}{\lambda^2}\sin^2(m\lambda)$$

$$+ 4F_0 \hat{g}_\sigma^\alpha(\lambda)\mathrm{Re}\left(\sum_{k=-m}^{m} \langle a_k\rangle e^{-i\lambda k}\right),$$

and remembering that $\langle a_k \rangle = 0$, the last equation transforms into

$$\langle|\hat{v}(\lambda)|^2\rangle = \lambda^2 |\hat{g}_\sigma^\alpha(\lambda)|^2 \left\langle \left|\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right|^2 \right\rangle$$

$$+ \frac{4F_0^2}{\lambda^2}\sin^2(m\lambda). \tag{A10}$$

The sum appearing in Eq. (A10) can be divided into two parts:

$$\left|\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right|^2 = \left(\sum_{k=-m}^{m} a_k e^{-i\lambda k}\right)\left(\sum_{l=-m}^{m} a_l e^{i\lambda l}\right)$$

$$= \sum_{k=-m}^{m} a_k^2 + \sum_{k \neq l}^{m} a_k a_l e^{i\lambda(l-k)}.$$

Averaging the last expression with the probability density function $P(a)$ leads us to the following result:

$$\begin{aligned}\left\langle \left| \sum_{k=-m}^{m} a_k e^{-i\lambda k} \right|^2 \right\rangle &= \sum_{k=-m}^{m} \langle a_k^2\rangle + \sum_{k\neq l}^{m} \langle a_k a_l\rangle e^{i\lambda(l-k)} \\ &= \langle a_k^2\rangle \sum_{k=-m}^{m} 1 + \langle a_k a_l\rangle \sum_{k\neq l}^{m} e^{i\lambda(l-k)} \\ &= (m+1)\Delta^2 + \langle a_k\rangle\langle a_l\rangle \sum_{k\neq l}^{m} e^{i\lambda(l-k)} \\ &= (m+1)\Delta^2,\end{aligned}$$

where we have used the statistical independence of the variables a_k to write $\langle a_k a_l\rangle = \langle a_k\rangle\langle a_l\rangle = 0$. Inserting the last result into Eq. (A10), we obtain

$$\langle|\hat{v}(\lambda)|^2\rangle = \lambda^2 |\hat{g}^{\alpha}_{\sigma}(\lambda)|^2 (m+1)\Delta^2 + \frac{4F_0^2}{\lambda^2}\sin^2(m\lambda). \tag{A11}$$

Note that the second term on the right hand side of Eq. (A11) is just the power spectrum of the constant force F given in Eq. (30). This term contributes to the overall power spectrum of the velocity just with a very sharp peak at $\lambda = 0$ (for $m \to \infty$ it is a Dirac delta function), and does not carry along any useful information on the potential structure. For this reason, this term has been omitted from the analysis in the main body of the work.

[1] J. M. Ziman, *Models of Disorder: The Theoretical Physics of Homogeneously Disordered Systems* (Cambridge University Press, Cambridge, England, 1979).

[2] A. H. Romero and J. M. Sancho, Phys. Rev. E **58**, 2833 (1998).

[3] A. Engel and F. Moss, Phys. Rev. A **38**, 571 (1988).

[4] S. O. Rice, Bell Syst. Tech. J. **24**, 41 (1945).

[5] P. W. Atkins, *Physical Chemistry*, 3rd ed. (Oxford University Press, London, 1986).

[6] If the charges a_k along the polymer are correlated with a *finite* correlation length ξ, we can always ‘‘break’’ the polymer into blocks of size ξ which are not correlated. We can then interpret each block as a new particle whose charge is the average of the charges inside the block, obtaining a situation in which the new block charges along the polymer are not correlated.

[7] William Feller, *An Introduction to Probability Theory and its Applications* 2nd ed. (John Wiley and Sons, New York, 1978), Vol. 2.

[8] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1989).

[9] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd ed. (Cambridge University Press, Cambridge, England, 1992).

[10] For the opposite limiting case when $\sigma \to 0$, dominant frequencies appear which are related to the potential disorder, as has been shown by M. Aldana, F. Cázarez-Bush, G. Cocho, and G. Martínez-Mekler, Physica A **257**, 119 (1998).